PRODUCT FIXED POINTS IN ORDERED METRIC SPACES

MIHAI TURINICI

Abstract. All product fixed point results in ordered metric spaces based on linear contractive conditions are but a vectorial form of the fixed point statement due to Nieto and Rodriguez-Lopez [Order, 22 (2005), 223-239], under the lines in Matkowski [Bull. Acad. Pol. Sci. (Ser. Sci. Math. Astronom. Phys.), 21 (1973), 323-324].

1. Introduction

Let $(X, d; \leq)$ be a partially ordered metric space; and $T : X \to X$ be a selfmap of X, with

(a01) $X(T, \leq) := \{x \in X; x \leq Tx\}$ is nonempty
(a02) T is increasing ($x \leq y$ implies $Tx \leq Ty$).

We say that $x \in X(T, \leq)$ is a *Picard point* (modulo $(d, \leq; T)$) if **i)** $(T^n x; n \geq 0)$ is d-convergent, **ii)** $z := \lim_n T^n x$ belongs to $\mathrm{fix}(T)$ (in the sense: $z = Tz$). If this happens for each $x \in X(T, \leq)$, then T is referred to as a *Picard operator* (modulo $(d, \leq)$); moreover, if these conditions hold for each $x \in X$, and **iii)** $\mathrm{fix}(T)$ is a singleton, then T is called a *strong Picard operator* (modulo $(d, \leq)$); cf. Rus [12, Ch 2, Sect 2.2]. Sufficient conditions for such properties are obtainable under metrical contractive requirements. Namely, call T, $(d, \leq; \alpha)$-*contractive* (where $\alpha > 0$), if

(a03) $d(Tx, Ty) \leq \alpha d(x, y)$, $\forall x, y \in X$, $x \leq y$.

Let $(x_n; n \geq 0)$ be a sequence in X; call it $(\leq)$-*ascending (descending)*, if $x_n \leq x_m$ ($x_n \geq x_m$), provided $n \leq m$. Further, let us say that $u \in X$ is an *upper (lower) bound* of this sequence, when $x_n \leq u$ ($x_n \geq u$), $\forall n$; if such elements exist, we say that $(x_n; n \geq 0)$ is *bounded above (below)*. Finally, call $(\leq)$, *d-self-closed* when the d-limit of each ascending sequence is an upper bound of it.

Theorem 1. *Assume that d is complete and T is $(d, \leq; \alpha)$-contractive, for some $\alpha \in]0, 1[$. If, in addition,*

(a04) *either T is d-continuous or $(\leq)$ is d-self-closed*

then, T is a Picard operator (modulo $(d, \leq)$). Moreover, if (in addition to (a04))

(a05) *for each $x, y \in X$, $\{x, y\}$ has a lower and upper bound*

then, T is a strong Picard one (modulo $(d, \leq)$).

Note that the former conclusion was obtained in 2005 by Nieto and Rodriguez-Lopez [7]; and the latter one is just the 2004 result in Ran and Reurings [11]. For appropriate extensions of these, we refer to Section 3 below.

2010 *Mathematics Subject Classification.* 47H10 (Primary), 54H25 (Secondary).
Key words and phrases. (Vector-valued) metric space, (quasi-) order, (product) fixed point, increasing map, completeness, continuity, self-closeness, normal matrix.

According to certain authors (cf. [8] and the references therein) these two results are the first extension of the Banach's contraction mapping principle to the realm of (partially) ordered metric spaces. However, the assertion is not entirely true: some early statements of this type have been obtained in 1986 by Turinici [13, Sect 2], in the context of ordered metrizable uniform spaces.

Now, these fixed point results found some useful applications to matrix and differential/integral equations theory; see the quoted papers for details. As a consequence, Theorem 1 was the subject of many extensions. Among these, we mention the coupled and tripled fixed point results in product ordered metric spaces, constructed under the lines in Bkaskar and Lakshmikantham [3]. It is our aim in the following to show that, for all such results based on "linear" contractive conditions, a reduction to Theorem 1 is possible; we refer to Section 4 and Section 5 for details. The basic tool is the concept of *normal matrix* due to Matkowski [6] (cf. Section 2). Further aspects will be delineated elsewhere.

2. Normal matrices

Let R^n denote the usual vector n-dimensional space, R^n_+ the standard positive cone in R^n, and $\leq$, the induced ordering. Also, let $(R^0_+)^n$ denote the interior of R^n and $<$ the strict (irreflexive transitive) ordering induced by it, in the sense

$(\xi_1, ...\xi_n) < (\eta_1, ..., \eta_n)$ provided $\xi_i < \eta_i$, $i \in \{1, ..., n\}$.

We shall indicate by $L(R^n)$ the (linear) space of all (real) $n \times n$ matrices $A = (a_{ij})$ and by $L_+(R^n)$ the positive cone of $L(R^n)$ consisting of all matrices $A = (a_{ij})$ with $a_{ij} \geq 0$, $i, j \in \{1, ..., n\}$. For each $A \in L_+(R^n)$, let us put

$\nu(A) = \inf\{\lambda \geq 0; Az \leq \lambda z, \text{ for some } z > 0\}$;

and call A, *normal*, if $\nu(A) < 1$; or, equivalently, when the system of inequalities

$$a_{i1}\zeta_1 + ... + a_{in}\zeta_n < \zeta_i, \quad i \in \{1, ..., n\} \tag{2.1}$$

has a solution $z = (\zeta_1, ..., \zeta_n) > 0$. Concerning the problem of characterizing this class of matrices, the following result obtained by Matkowski [6] must be taken into consideration. Denote (for $1 \leq i, j \leq n$)

(b01) $a^{(1)}_{ij} = 1 - a_{ij}$ if $i = j$; $a^{(1)}_{ij} = a_{ij}$, if $i \neq j$;

and, inductively (for $1 \leq k \leq n - 1$, $k + 1 \leq i, j \leq n$)

(b02) $a^{(k+1)}_{ij} = a^{(k)}_{kk} a^{(k)}_{ij} - a^{(k)}_{ik} a^{(k)}_{kj}$, if $i = j$; $a^{(k+1)}_{ij} = a^{(k)}_{kk} a^{(k)}_{ij} + a^{(k)}_{ik} a^{(k)}_{kj}$, if $i \neq j$.

Proposition 1. *The matrix $A \in L_+(R^n)$ is normal, if and only if*

(b03) $a^{(i)}_{ii} > 0, \quad i \in \{1, ..., n\}$.

Proof. Necessity. Assume that (2.1) has a solution $z = (\zeta_1, ..., \zeta_n) > 0$; that is

$$\begin{array}{rrrrrl} a^{(1)}_{11}\zeta_1 & -a^{(1)}_{12}\zeta_2 & -a^{(1)}_{13}\zeta_3 & -... & -a^{(1)}_{1n}\zeta_n & > 0 \\ -a^{(1)}_{21}\zeta_1 & +a^{(1)}_{22}\zeta_2 & -a^{(1)}_{23}\zeta_3 & -... & -a^{(1)}_{2n}\zeta_n & > 0 \\ -a^{(1)}_{31}\zeta_1 & -a^{(1)}_{32}\zeta_2 & +a^{(1)}_{33}\zeta_3 & -... & -a^{(1)}_{3n}\zeta_n & > 0 \\ ... & ... & ... & ... & ... & ... \\ -a^{(1)}_{n1}\zeta_1 & -a^{(1)}_{n2}\zeta_2 & -a^{(1)}_{n3}\zeta_3 & -... & +a^{(1)}_{nn}\zeta_n & > 0. \end{array} \tag{2.2}$$

In view of

$$a^{(1)}_{ij} \geq 0, \quad i, j \in \{1, ..., n\}, \quad i \neq j, \tag{2.3}$$

we must have (by these conditions) $a_{11}^{(1)},...,a_{nn}^{(1)}>0$; hence, in particular, (b03) is fulfilled for $i=1$. Further, let us multiply the first inequality of (2.2) by the factor $a_{i1}^{(1)}/a_{11}^{(1)}\geq 0$ and add it to the i-th relation of the same system for $i\in\{2,...,n\}$; one gets [if we take into account (b02) (for $k=1$) plus $a_{11}^{(1)}>0$]

$$\begin{array}{cccccc}
a_{11}^{(1)}\zeta_1 & -a_{12}^{(1)}\zeta_2 & -a_{13}^{(1)}\zeta_3 & -... & -a_{1n}^{(1)}\zeta_n & >0\\
 & a_{22}^{(2)}\zeta_2 & -a_{23}^{(2)}\zeta_3 & -... & -a_{2n}^{(2)}\zeta_n & >0\\
 & -a_{32}^{(2)}\zeta_2 & +a_{33}^{(2)}\zeta_3 & -... & -a_{3n}^{(2)}\zeta_n & >0\\
 & ... & ... & ... & ... & ...\\
 & -a_{n2}^{(2)}\zeta_2 & -a_{n3}^{(2)}\zeta_3 & -... & +a_{nn}^{(2)}\zeta_n & >0.
\end{array} \tag{2.4}$$

Since (see above)

$$a_{ij}^{(2)}\geq 0,\quad i,j\in\{2,...,n\},\quad i\neq j, \tag{2.5}$$

we must have (by these conditions) $a_{22}^{(2)},...,a_{nn}^{(2)}>0$; wherefrom, (b03) is fulfilled for $i\in\{1,2\}$. Now, if we multiply the second inequality of (2.4) by the factor $a_{i2}^{(2)}/a_{22}^{(2)}\geq 0$ and add it to the i-th relation of the same system for $i\in\{3,...,n\}$, one obtains that (b03) will be fulfilled with $i\in\{1,2,3\}$; and so on. Continuing in this way, it is clear that, after n steps, (b03) will be entirely satisfied.

Sufficiency. Assume that (b03) holds; we must find a solution $z=(\zeta_1,...,\zeta_n)$ for (2.1) with $\zeta_i>0$, $i\in\{1,...,n\}$. To do this, let us start from the system

$$\begin{array}{cccccc}
a_{11}^{(1)}\zeta_1 & -a_{12}^{(1)}\zeta_2 & -a_{13}^{(1)}\zeta_3 & -... & -a_{1n}^{(1)}\zeta_n & =\sigma_1\\
-a_{21}^{(1)}\zeta_1 & +a_{22}^{(1)}\zeta_2 & -a_{23}^{(1)}\zeta_3 & -... & -a_{2n}^{(1)}\zeta_n & =\sigma_2\\
-a_{31}^{(1)}\zeta_1 & -a_{32}^{(1)}\zeta_2 & +a_{33}^{(1)}\zeta_3 & -... & -a_{3n}^{(1)}\zeta_n & =\sigma_3\\
... & ... & ... & ... & ... & ...\\
-a_{n1}^{(1)}\zeta_1 & -a_{n2}^{(1)}\zeta_2 & -a_{n3}^{(1)}\zeta_3 & -... & +a_{nn}^{(1)}\zeta_n & =\sigma_n
\end{array} \tag{2.6}$$

where $y=(\sigma_1,...,\sigma_n)>0$ is arbitrary fixed. Denote

(b04) $\sigma_i^{(1)}=\sigma_i$ (hence $\sigma_i^{(1)}>0$), $i\in\{1,...,n\}$;

and, inductively (for $1\leq k\leq n-1$)

(b05) $\sigma_i^{(k+1)}=a_{kk}^{(k)}\sigma_i^{(k)}+a_{ik}^{(k)}\sigma_k^{(k)}$, $k+1\leq i\leq n$.

Let us apply to (2.6) the same transformations as in (2.2); one gets

$$\begin{array}{cccccc}
a_{11}^{(1)}\zeta_1 & -a_{12}^{(1)}\zeta_2 & -a_{13}^{(1)}\zeta_3 & -... & -a_{1n}^{(1)}\zeta_n & =\sigma_1^{(1)}\\
 & a_{22}^{(2)}\zeta_2 & -a_{23}^{(2)}\zeta_3 & -... & -a_{2n}^{(1)}\zeta_n & =\sigma_2^{(2)}\\
 & -a_{32}^{(2)}\zeta_2 & +a_{33}^{(2)}\zeta_3 & -... & -a_{3n}^{(2)}\zeta_n & =\sigma_3^{(2)}\\
 & ... & ... & ... & ... & ...\\
 & -a_{n2}^{(2)}\zeta_2 & -a_{n3}^{(2)}\zeta_3 & -... & +a_{nn}^{(2)}\zeta_n & =\sigma_n^{(2)};
\end{array} \tag{2.7}$$

where, in addition (taking (2.3) into account)

$$\sigma_i^{(2)}>0,\quad i\in\{2,...,n\}. \tag{2.8}$$

If we apply to this new system the same transformations as in (2.4) and, further, iterate these upon the obtained system, etc., we arrive at

$$\begin{array}{cccccl} a_{11}^{(1)}\zeta_1 & -a_{12}^{(1)}\zeta_2 & -a_{13}^{(1)}\zeta_3 & -... & -a_{1n}^{(1)}\zeta_n & =\sigma_1^{(1)} \\ & a_{22}^{(2)}\zeta_2 & -a_{23}^{(2)}\zeta_3 & -... & -a_{2n}^{(1)}\zeta_n & =\sigma_2^{(2)} \\ & & a_{33}^{(3)}\zeta_3 & -... & -a_{3n}^{(3)}\zeta_n & =\sigma_3^{(3)} \\ & & & ... & ... & ... \\ & & & & a_{nn}^{(n)}\zeta_n & =\sigma_n^{(n)} \end{array} \tag{2.9}$$

(the diagonal form of (2.6)). From (b03) [plus the positivity properties of type (2.8)], the unique solution $z = (\zeta_1, ..., \zeta_n)$ of (2.9) satisfies $\zeta_i > 0$, $i \in \{1, ..., n\}$; this, and the equivalence between (2.6) and (2.9), ends the argument. $\square$

A useful variant of Matkowski's condition (b03) may now be depicted as follows. Letting I denote the unitary matrix in $L(R^n)$, indicate by Δ_1,...,Δ_n the successive "diagonal" minors of $I - A$; that is

$$\Delta_1 = 1 - a_{11},\ \Delta_2 = \det\begin{pmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22}\end{pmatrix}, ..., \Delta_n = \det(I - A).$$

By the transformations we used in passing from (2.6 to (2.7) and from this to the next one, etc., one gets $\Delta_i = a_{ii}^{(i)}$, $1 \le i \le n$; so that, (b03) writes

(b06) $\Delta_i > 0$, $i \in \{1, ..., n\}$.

After Perov's terminology [9], a matrix $A \in L_+(R^n)$ satisfying (b06) will be termed *admissible* (or, equivalently: a-matrix). We therefore proved

Proposition 2. *Over the subclass $L_+(R^n)$, we have normal $\Longleftrightarrow$ admissible.*

For practical and theoretical reasons, further characterizations of this class of matrices are necessary. To this end, let $||.||$ denote one of the usual p-norms in R^n (where $1 \le p \le \infty$), introduced as: for $x = (\xi_1, ..., \xi_n) \in R^n$,

$$||x||_p = (|\xi_1|^p + ... + |\xi_n|^p)^{1/p}\ (1 \le p < \infty);\ ||x||_\infty = \max\{|\xi_1|, ..., |\xi_n|\}.$$

Note that, all these have the important property

$$x, y \in R_+^n,\ x \le y \Longrightarrow ||x|| \le ||y|| \quad (||.|| \text{ is } monotone). \tag{2.10}$$

Let also $||.||^*$ stand for the compatible matrix norm in $L(R^n)$:

(b07) $||A||^* = \sup\{||Ax||; ||x|| \le 1\}$, $A \in L(R^n)$.

We stress that, for the arbitrary fixed $p \in [1, \infty]$, the compatible to $||.||_p$ matrix norm $||.||_p^*$ in $L(R^n)$ is not identical with the p-norm of $L(R^n)$, obtained by a formal identification of it with $R^{(n^2)}$. For example, one has

$$||A||_1^* = \max\{||Ae_1||_1, ..., ||Ae_n||_1\},\ ||A||_\infty^* = \max\{||e_1^\top A||_1, ..., ||e_n^\top A||_1\}; \tag{2.11}$$

where $E = (e_1, ..., e_n)$ is the canonical basis of R^n; hence the claim.

In the following, it will be convenient to take $(||.||; ||.||^*)$ as $(||.||_1; ||.||_1^*)$.

Remark 1. Let $||.||_V$ be an arbitrary norm of R^n; and $||.||_V^*$, its associated matrix norm in $L(R^n)$. By a well known result (see, for instance, Precupanu [10, Ch 2, Sect 2.5]) $||.||_V$ and $||.||$ (as defined above) are equivalent:

$$\beta||x|| \le ||x||_V \le \gamma||x||,\ \forall x \in R^n, \text{ where } 0 < \beta < \gamma;$$

and so are the compatible matrix norms $||.||_V^*$ and $||.||^*$. A direct verification of this last fact is to be obtained by means of (b07); we do not give details.

Having these precise, call $A \in L(R^n)$, *asymptotic* provided

$A^p \to 0$ (in the matrix norm $||.||^*$) as $p \to \infty$;

or, equivalently (see (2.11)) if it fulfills one of the properties

$A^p x \to 0$ as $p \to \infty$, $\forall x \in R^n_+$; $A^p x \to 0$ as $p \to \infty$, $\forall x \in R^n$.

The following simple result will be in effect for us.

Lemma 1. *For the matrix $A \in L_+(R^n)$ we have*

$$[A \text{ is asymptotic}] \iff \text{the series } \sum_{p\geq 0} A^p \text{ converges in } (L(R^n), ||.||^*). \tag{2.12}$$

In such a case, the sum of this series is $(I-A)^{-1}$; hence, $I-A$ is invertible in $L(R^n)$ and its inverse belongs to $L_+(R^n)$.

Proof. Let the matrix A be asymptotic. If $x \in R^n$ satisfies $(I-A)x = 0$ then, (by repeatedly applying A to the equivalent equality) $x = A^p x$, for all $p \in N$; wherefrom $x = 0$ (if one takes the limit as $p \to \infty$); proving that $(I-A)^{-1}$ exists as an element of $L(R^n)$. Moreover, in view of $I - A^p = (I-A)(I + A + ... + A^{p-1})$, $p > 1$, one gets (again by a limit process) $I = (I-A)(I + A + A^2 + ...)$; which ends the argument. □

As before, we may ask of which relationships exist between this class of matrices and the preceding ones. To do this, the following renorming statement involving normal matrices will be useful.

Lemma 2. *Let $A \in L_+(R^n)$ be a normal matrix. Then, an equivalent monotonic norm $||.||_A$ in R^n and a number α in $]0,1[$ exist with the property*

$$||Ax||_A \leq \alpha ||x||_A, \text{ for all } x \in R^n_+. \tag{2.13}$$

Proof. By hypothesis, we have promised a vector $z = (\zeta_1, ..., \zeta_n) > 0$ and a number α in $]\nu(A), 1[$ with $Az \leq \alpha z$. Define a norm $||.||_A$ in R^n as

(b08) $||x||_A = \max\{|\xi_i|/\zeta_i; 1 \leq i \leq n\}$, $x = (\xi_1, ..., \xi_n) \in R^n$.

By the obvious relation

$$x \leq (||x||_A)z, \text{ for all } x \in R^n_+, \tag{2.14}$$

one gets (if we take into account the choice of z)

$$Ax \leq (||x||_A)Az \leq (\alpha ||x||_A)z, \quad x \in R^n_+;$$

wherefrom, (2.13) results. Since the monotonic property is evident, we omit the details. It remains only to prove that $||.||_A$ is equivalent with the initial norm $||.||$ in R^n. But this follows easily by the relation (deduced from (b08) and (2.14))

$$\beta ||x||_A \leq ||x|| \leq \gamma ||x||_A, \quad x \in R^n; \tag{2.15}$$

where $\beta = \min\{\zeta_i; 1 \leq i \leq n\}$, $\gamma = \zeta_1 + ... + \zeta_n$. The proof is complete. □

We may now give an appropriate answer to the above posed problem.

Proposition 3. *For each matrix of $L_+(R^n)$, we have normal $\iff$ asymptotic.*

Proof. Let $A \in L_+(R^n)$ be normal. By Lemma 2, we found an equivalent monotonic norm $||.||_A$ on R^n, and an $\alpha \in]0,1[$, with the property (2.13). From this, we get

$$A^p x \to 0 \text{ (modulo } ||.||_A) \text{ as } p \to \infty, \text{ for all } x \in R^n_+,$$

which, according to (2.15), is just the asymptotic property. Conversely, let $A \in L_+(R^n)$ be asymptotic. Fix $b = (\beta_1, ..., \beta_n) > 0$ and put $z = \sum_{p\geq 0} A^p b$ (hence,

$z > 0$). As $Az = \sum_{p\geq 1} A^p b$, we have $z = b + Az$; which, combined with the choice of b, gives $Az < z$. □

We cannot close these developments without giving another characterization of asymptotic (or normal) matrices in terms of *spectral radius*; this fact – of marginal importance for the next section – is, however, sufficiently interesting by itself to be added here. Let $A \in L(R^n)$ be a matrix. Under the natural immersion of R^n in C^n, let us call the number $\lambda \in C$ an *eigenvalue* of A, provided $Az = \lambda z$, for some different from zero vector $z \in C^n$ (called in this case an *eigenvector* of A); the class of all these numbers will be denoted $\sigma(A)$ (the *spectrum* of A). Define

$$\rho(A) = \sup\{|\lambda|; \lambda \in \sigma(A)\} \quad \text{(the \textit{spectral radius} of } A).$$

Proposition 4. *The matrix $A \in L_+(R^n)$ is asymptotic if and only if $\rho(A) < 1$.*

Proof. Suppose A is asymptotic. For each eigenvalue $\lambda \in \sigma(A)$, let $z = z(\lambda) \in C^n$ be the corresponding eigenvector of A. We have $Az = \lambda z$; and this gives $A^p z = \lambda^p z$, for all $p \in N$. By the choice of A plus $z \neq 0$, we must have $\lambda^p \to 0$ as $p \to \infty$, which cannot happen unless $|\lambda| < 1$; hence $\rho(A) < 1$. Conversely, assume that the matrix $A = (a_{ij})$ in $L_+(R^n)$ satisfies $\rho(A) < 1$; and put $A^{(\varepsilon)} = (a_{ij}^{(\varepsilon)})$, $\varepsilon > 0$, where

$$a_{ij}^{(\varepsilon)} = a_{ij} + \varepsilon, \quad 1 \leq i, j \leq n.$$

We have $\rho(A^{(\varepsilon)}) < 1$, when $\varepsilon > 0$ is small enough (one may follow a direct argument based on the obvious fact: for each (nonempty) compact K of R,

$$\det(\lambda I - A^{(\varepsilon)}) \to \det(\lambda I - A) \text{ when } \varepsilon \to 0+, \text{ uniformly over } \lambda \in K;$$

we do not give further details). Now, as $A^{(\varepsilon)}$ is a matrix over R_+^0 (in the sense: $a_{ij}^{(\varepsilon)} > 0$, $i, j \in \{1, ..., n\}$), we have, by the Perron-Frobenius theorem (see, e.g., Bushell [4]), that for a sufficiently small $\varepsilon > 0$, $A^{(\varepsilon)}$ has a positive eigenvalue $\mu = \mu(\varepsilon) > 0$ (which, in view of $\rho(A^{(\varepsilon)}) < 1$, must satisfy $\mu < 1$), as well as a corresponding eigenvector $z = z(\varepsilon) > 0$. But then, $Az \leq A^{(\varepsilon)} z = \mu z < z$; hence, A is normal. This, along with Proposition 3, completes the argument. □

3. Extension of Theorem 1

Let (X, d) be a metric space; and $(\leq)$ be a it quasi-order (i.e.: reflexive and transitive relation) over X. For each $x, y \in X$, denote: $x <> y$ iff either $x \leq y$ or $y \leq x$ (i.e.: x and y are comparable). This relation is reflexive and symmetric; but not in general transitive. Given $x, y \in X$, any subset $\{z_1, ..., z_k\}$ (for $k \geq 2$) in X with $z_1 = x$, $z_k = y$, and $[z_i <> z_{i+1},\ i \in \{1, ..., k-1\}]$ will be referred to as a $<>$-*chain* between x and y; the class of all these will be denoted as $C(x, y; <>)$. Let $\sim$ stand for the relation (over X): $x \sim y$ iff $C(x, y; <>)$ is nonempty. Clearly, $(\sim)$ is reflexive and symmetric; because so is $<>$. Moreover, $(\sim)$ is transitive; hence, it is an equivalence over X. Call d, $(\leq)$-*complete* when each ascending d-Cauchy sequence is d-convergent. Finally, let $T : X \to X$ be a selfmap of X; we say that it is $(d, \leq)$-*continuous* when [(x_n)=ascending, $x_n \xrightarrow{d} x$] imply $Tx_n \xrightarrow{d} Tx$.

Theorem 2. *Assume (under (a01) and (a02)) that d is $(\leq)$-complete and T is $(d, \leq; \alpha)$-contractive, for some $\alpha \in]0, 1[$. If, in addition,*

(c01) *either T is $(d, \leq)$-continuous or $(\leq)$ is d-self-closed,*

then, T is a Picard operator (modulo $(d, \leq)$). Moreover, if (in addition to (c01))

(c02) $(\sim) = X \times X$ *[$C(x, y; <>)$ is nonempty, for each $x, y \in X$],*
then, T is a strong Picard operator (modulo $(d, \leq)$).

This result is a weaker form of Theorem 1; because (a04) $\Longrightarrow$ (c01), (a05) $\Longrightarrow$ (c02). [In fact, given $x, y \in X$, there exist, by (a05), some $u, v \in X$ with $u \leq x \leq v$, $u \leq y \leq v$. This yields $x <> u$, $u <> y$; wherefrom, $x \sim y$]. Its proof mimics, in fact, the one of Theorem 1. However, for completeness reasons, we shall provide it, with some modifications.

Proof. **I)** Let $x \in X(T, \leq)$ be arbitrary fixed; and put $x_n = T^n x$, $n \in N$. By (a02) and (a03), $d(x_{n+1}, x_{n+2}) \leq \alpha d(x_n, x_{n+1}))$, for all n. This yields $d(x_n, x_{n+1}) \leq \alpha^n d(x_0, x_1))$, $\forall n$; so that, as the series $\sum_n \alpha^n$ converges, $(x_n; n \geq 0)$ is an ascending d-Cauchy sequence. Combining with the $(\leq)$-completeness of d, it results that $x_n \xrightarrow{d} x^*$, for some $x^* \in X$. Now, if the first half of (c01) holds, we have $x_{n+1} = Tx_n \xrightarrow{d} Tx^*$; so that (as d=metric), $x^* \in \text{fix}(T)$. Suppose that the second half of (c01) is valid; note that, as a consequence, $x_n \leq x^*$, $\forall n$. By the contractive condition, we derive $d(x_{n+1}, Tx^*) \leq \alpha d(x_n, x^*)$, $\forall n$; so that, by the obtained convergence property, $x_{n+1} = Tx_n \xrightarrow{d} Tx^*$; wherefrom (see above) $x^* \in \text{fix}(T)$.

II) Take $a, b \in X$, $a \leq b$. By the contractive condition, $d(T^n a, T^n b) \leq \alpha^n d(a, b)$, $\forall n$; whence $\lim_n d(T^n a, T^n b) = 0$. From the properties of the metric, one gets $\lim_n d(T^n a, T^n b) = 0$ if $a <> b$; as well as (by definition) $\lim_n d(T^n a, T^n b) = 0$ if $a \sim b$. This, along with (c02), gives the desired conclusion. $\square$

4. Vector linear contractions

Let X be an abstract set; and $q \geq 1$ be a positive integer. In the following, the notion of *R^q-valued metric* on X will be used to designate any function $\Delta : X^2 \to R^q_+$, supposed to be *reflexive sufficient* [$\Delta(x, y) = 0$ iff $x = y$] *triangular* [$\Delta(x, z) \leq \Delta(x, y) + \Delta(y, z)$, $\forall x, y, z \in X$] and *symmetric* [$\Delta(x, y) = \Delta(y, x)$, $\forall x, y \in X$]. In this case, the couple (X, Δ) will be termed an *R^q-valued metric space*. Fix in the following such an object; as well the usual norm $||.|| := ||.||_1$, over R^q. Note that, in such a case, the map

(d01) $(d : X^2 \to R_+)$: $d(x, y) = ||\Delta(x, y)||$, $x, y \in X$

is a (standard) metric on X. Let also $(\preceq)$ be a *quasi-ordering* over X.

Define a Δ-convergence property over X as: [$x_n \xrightarrow{\Delta} x$ iff $\Delta(x_n, x) \to 0$]. The set of all such x will be denoted $\lim_n(x_n)$; when it is nonempty (hence, a singleton), (x_n) will be termed *Δ-convergent*. Further, call (x_n), *Δ-Cauchy* provided [$\Delta(x_i, x_j) \to 0$ as $i, j \to \infty$]. Clearly, each Δ-convergent sequence is Δ-Cauchy; but the converse is not general valid. Note that, in terms of the associated metric d,

$$[\forall (x_n), \forall x]\text{: } x_n \xrightarrow{\Delta} x \text{ iff } x_n \xrightarrow{d} x \tag{4.1}$$

$$[\forall (x_n)]\text{: } (x_n) \text{ is } \Delta\text{-Cauchy iff } (x_n) \text{ is } d\text{-Cauchy.} \tag{4.2}$$

Call Δ, *$(\preceq)$-complete* when each ascending Δ-Cauchy sequence is Δ-convergent. Likewise, call $(\preceq)$, *Δ-self-closed* when the Δ-limit of each ascending sequence is an upper bound of it. By (4.1) and (4.2) we have the global properties

$$[\Delta \text{ is } (\preceq)\text{-complete] iff } [d \text{ is } (\preceq)\text{-complete}] \tag{4.3}$$

$$[(\preceq) \text{ is } \Delta\text{-self-closed] iff } [(\preceq) \text{ is } d\text{-self-closed}]. \tag{4.4}$$

Finally, take a selfmap $T : X \to X$, according to

(d02) $X(T, \preceq) := \{x \in X; x \preceq Tx\}$ is nonempty
(d03) T is increasing ($x \preceq y$ implies $Tx \preceq Ty$).

We say that $x \in X(T, \preceq)$ is a *Picard point* (modulo $(\Delta, \preceq; T)$) if **j)** $(T^n x; n \geq 0)$ is Δ-convergent, **jj)** $z := \lim_n(T^n x)$ belongs to $\mathrm{fix}(T)$. If this happens for each $x \in X(T, \preceq)$, then T is referred to as a *Picard operator* (modulo $(\Delta, \preceq)$). Sufficient conditions for such properties are to be obtained under vectorial contractive requirements. Given $A \in L_+(R^q)$, let us say that T is $(\Delta, \preceq; A)$*-contractive*, provided

(d04) $\Delta(Tx, Ty) \leq A\Delta(x, y), \ \ \forall x, y \in X, \ x \preceq y$.

Further, let us say that T is $(\Delta, \preceq)$*-continuous* when [(x_n)=ascending and $x_n \xrightarrow{\Delta} x$] imply $Tx_n \xrightarrow{\Delta} Tx$. As before, in terms of the associated via (d01) metric d, we have (by means of (4.1) and (4.2) above)

$$[T \text{ is } (\Delta, \preceq)\text{-continuous}] \text{ iff } [T \text{ is } (d, \preceq)\text{-continuous}]. \tag{4.5}$$

The following answer to the posed question is available.

Theorem 3. *Assume (under (d02) and (d03)) that Δ is $(\preceq)$-complete and there exists a normal $A \in L_+(R^q)$ such that T is $(\Delta, \preceq; A)$-contractive. In addition, suppose that*

(d05) *either (T is $(\Delta, \preceq)$-continuous) or ($(\preceq)$ is Δ-self-closed).*

Then, T is a Picard operator (modulo $(\Delta, \preceq)$).

Proof. As A is normal, there exist, by Lemma 2, an equivalent (with $||.||$) monotonic norm $||.||_A$ on R^q, and an $\alpha \in]0, 1[$, fulfilling (2.13). Define a new metric $e(.,.)$ over X, according to

(d06) $e(x, y) = ||\Delta(x, y)||_A, \quad x, y \in X$.

By the norm equivalence (2.15), the properties (4.1)-(4.5) written in terms of d continue to hold in terms of e. Moreover, by the monotonic property and (2.13),

$$e(Tx, Ty) \leq \alpha e(x, y), \ \ \forall x, y \in X, \ x \preceq y. \tag{4.6}$$

Summing up, Theorem 2 is applicable to $(X, e; \preceq)$ and T; wherefrom, all is clear. □

In particular, when $(\preceq) = X^2$ (the *trivial quasi-order* on X) the corresponding version of Theorem 3 is just the statement in Perov [9].

5. Product fixed points

Let $\{(X_i, d_i; \leq_i); 1 \leq i \leq q\}$ be a system of quasi-ordered metric spaces. Denote $X = \prod\{X_i; 1 \leq i \leq q\}$ (the Cartesian product of the ambient sets); and put, for $x = (x_1, ..., x_q)$ and $y = (y_1, ..., y_q)$ in X

(e01) $\Delta(x, y) = (d_1(x_1, y_1), ..., d_q(x_q, y_q))$,
(e02) $x \preceq y$ iff $x_i \leq_i y_i$, $i \in \{1, ..., q\}$.

Clearly, Δ is a R^q-valued metric on X; and $(\preceq)$ acts as a quasi-ordering over the same. As a consequence of this, we may now introduce all conventions in Section 4. Note that, by the very definitions above, we have, for the sequence $(x^n = (x_1^n, ..., x_q^n); n \geq 0)$ in X and the point $x = (x_1, ..., x_q)$ in X,

$$x^n \xrightarrow{\Delta} x \text{ iff } d_i(x_i^n, x_i) \to 0 \text{ as } n \to \infty, \text{ for all } i \in \{1, ..., q\} \tag{5.1}$$

$$(x^n; n \geq 0) \text{ is } \Delta\text{-Cauchy iff } (x_i^n; n \geq 0) \text{ is } d_i\text{-Cauchy}, \forall i \in \{1, ..., q\}. \tag{5.2}$$

This yields the useful global implications

$$[d_i \text{ is } (\le_i)\text{-complete}, \forall i \in \{1,...,q\}] \Longrightarrow \Delta \text{ is } (\preceq)\text{-complete} \tag{5.3}$$

$$[(\le_i) \text{ is } d_i\text{-self-closed}, \forall i \in \{1,...,q\}] \Longrightarrow (\preceq) \text{ is } \Delta\text{-self-closed}. \tag{5.4}$$

(I) We are now passing to our effective part. Let $(T_i : X \to X_i; 1 \le i \le q)$ be a system of maps; it generates an *associated* selfmap (of X)

(e03) $T : X \to X$: $Tx = (T_1x, ..., T_qx)$, $x = (x_1, ..., x_q) \in X$.

Suppose that

(e04) $\exists a = (a_1, ..., a_q) \in X : \ a_i \le_i T_i a, \ i \in \{1, ..., q\}$
(e05) T_i is increasing ($x \preceq y \Longrightarrow T_i x \le_i T_i y$), $i \in \{1, ..., q\}$.

Note that, as a consequence, (d02) and (d03) hold. For $i \in \{1, ..., q\}$, call T_i, $(\Delta, \preceq)$-*continuous*, when: $[(x^n = (x_1^n, ..., x_q^n))$=ascending and $x^n \overset{\Delta}{\longrightarrow} x]$ imply $d_i(T_i x^n, T_i x) \to 0$ as $n \to \infty$. Clearly,

$$[T_i \text{ is } (\Delta, \preceq)\text{-continuous}, i \in \{1, ..., q\}] \text{ implies } T \text{ is } (\Delta, \preceq)\text{-continuous}. \tag{5.5}$$

Let $A = (a_{ij}; 1 \le i, j \le q)$ be an element of $L_+(R^q)$. For $i \in \{1, ..., q\}$, denote $A_i = (a_{i1}, ..., a_{iq})$ (the i-th line of A). Call T_i, $(\Delta, \preceq; A_i)$-contractive, provided

(e06) $d_i(T_i x, T_i y) \le A_i \Delta(x, y), \ \forall x, y \in X, \ x \preceq y$.

The following implication is evident:

$$[T_i \text{ is } (\Delta, \preceq; A_i)\text{-contractive}, i \in \{1, ..., q\}] \Longrightarrow T \text{ is } (\Delta, \preceq; A)\text{-contractive}. \tag{5.6}$$

Putting these together, we have (via Theorem 3 above):

Theorem 4. *Assume (under (e04) and (e05)) that d_i is $(\le_i)$-complete, $\forall i \in \{1, ..., q\}$, and there exists a normal matrix $A = (A_1, ..., A_q)^\top \in L_+(R^q)$ such that T is $(\Delta, \preceq; A_i)$-contractive, $\forall i \in \{1, ..., q\}$. In addition, suppose that*

(e07) *either (T_i is $(\Delta, \preceq)$-continuous, $\forall i \in \{1, ..., q\}$)*
or ($(\le_i)$ is d_i-self-closed, $\forall i \in \{1, ..., q\}$).

Then, the associated selfmap T is a Picard one (modulo $(\Delta, \preceq)$).

In particular, when $(\le_i) = X_i \times X_i$, $i \in \{1, ..., q\}$, this result is just the one in Matkowski [6]. Some "uniform" versions of it were obtained in Czerwik [5]; see also Balakrishna Reddy and Subrahmanyam [1].

(II) By definition, any fixed point of the associated selfmap T will be referred to as a *product fixed point* of the original system $(T_1, ..., T_q)$. To see its usefulness, it will suffice noting that, by an appropriate choice of our data, one gets (concrete) coupled and tripled fixed point results in the area, obtainable via "linear" type contractive conditions. The most elaborated one, due to Berinde and Borcut [2] will be discussed below.

Let $(X, d; \le)$ be a partially ordered metric space; and take a map $F : X^3 \to X$. We say that $b = (b_1, b_2, b_3) \in X^3$ is a *tripled* fixed point of F, provided

(e08) $b_1 = F(b_1, b_2, b_3)$, $b_2 = F(b_2, b_1, b_2)$, $b_3 = F(b_3, b_2, b_1)$.

Sufficient conditions for the existence of such points are centered on

(e09) there exists at least one $a = (a_1, a_2, a_3) \in X^3$ with
$a_1 \le F(a_1, a_2, a_3)$, $a_2 \ge F(a_2, a_1, a_2)$, $a_3 \le F(a_3, a_2, a_1)$

(e10) F is *mixed monotone*:
$x_1 \le x_2 \Longrightarrow F(x_1, y, z) \le F(x_2, y, z)$,
$y_1 \le y_2 \Longrightarrow F(x, y_1, z) \ge F(x, y_2, z)$,
$z_1 \le z_2 \Longrightarrow F(x, y, z_1) \le F(x, y, z_2)$.

Let $(\preceq)$ be the ordering on X^3 introduced as

$(x_1, x_2, x_3) \preceq (y_1, y_2, y_3)$ iff $x_1 \le y_1$, $x_2 \ge y_2$, $x_3 \le y_3$.

Call F, $(d, \preceq; \alpha_1, \alpha_2, \alpha_3)$-contractive (where $\alpha_1, \alpha_2, \alpha_3 > 0$) when

(e11) $d(F(x_1, x_2, x_3), F(y_1, y_2, y_3)) \le \alpha_1 d(x_1, y_1) + \alpha_2 d(x_2, y_2) + \alpha_3 d(x_3, y_3)$, for all $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ in X^3 with $x \preceq y$.

Theorem 5. *Suppose (under (e09) and (e10)) that d is a complete metric and F is $(d, \preceq; \alpha_1, \alpha_2, \alpha_3)$-contractive, for some $\alpha_1, \alpha_2, \alpha_3 > 0$ with $\alpha := \alpha_1 + \alpha_2 + \alpha_3 < 1$. If, in addition, either [F is continuous] or [both $(\le)$ and $(\ge)$ are d-self-closed] then F has at least one tripled fixed point.*

See the quoted paper for the original argument. Here, we shall develop a different one, based on the fact that, any tripled fixed point for F is a fixed point of the associated selfmap T of X^3, introduced as:

(e12) $Tx = (F(x_1, x_2, x_3), F(x_2, x_1, x_2), F(x_3, x_2, x_1))^\top$, $x = (x_1, x_2, x_3) \in X^3$.

To do this, it will suffice verifying that conditions of Theorem 4 are fulfilled with $(X_1, d_1; \le_1) = (X, d; \le)$, $(X_2, d_2; \le_2) = (X, d; \ge)$, $(X_3, d_3; \le_3) = (X, d; \le)$.

Proof. (**Theorem 5**) Define, for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in X^3,

$\Delta(x, y) = (d(x_1, y_1), d(x_2, y_2), d(x_3, y_3))$.

We have to establish that the associated map T introduced via (e12) is increasing (modulo $(\preceq)$) and $(\Delta, \preceq; A)$-contractive, for a normal matrix $A \in L_+(R^3)$, The former of these is directly obtainable by means of the mixed monotone property (e10). For the latter, note that, by (e11), T is $(\Delta, \preceq; A)$-contractive, where $A \in L_+(R^3)$ is given as $A = (A_1, A_2, A_3)^\top$, where

$$A_1 = (\alpha_1, \alpha_2, \alpha_3), A_2 = (\alpha_2, \alpha_1 + \alpha_3, 0), A_3 = (\alpha_3, \alpha_2, \alpha_1).$$

Since, on the other hand, $A\Theta = \alpha\Theta < \Theta$, where $\Theta = (1, 1, 1)^\top$, it results that A is normal (cf. Section 2); and we are done. □

Remark 2. The last part of the argument above suggests us a simplified proof of the original Berinde-Borcut argument. Namely, given $(X, d; \le)$, F and $(\alpha_1, \alpha_2, \alpha_3)$ as in Theorem 5, define a standard metric $D(.,.)$ over X^3 as: for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in X^3,

(e13) $D(x, y) = \max\{d(x_1, y_1), d(x_2, y_2), d(x_3, y_3)\}$.

By the contractive condition (e11), it is clear that

$$D(Tx, Ty) \le \alpha D(x, y), \text{ for all } x, y \in X^3,\ x \preceq y. \tag{5.7}$$

This, along with the previous remarks, tells us that Theorem 1 applies to the ordered metric space $(X^3, D; \preceq)$ and T; wherefrom, all is clear.

Note, finally, that the original coupled fixed point statement in Bhaskar and Lakshmikantham [3] corresponds to the normal matrix (where $0 < \alpha < 1$)

$$A = (A_1, A_2)^\top \in L_+(R^2): \quad A_1 = A_2 = (\alpha/2)(1, 1).$$

Further aspects will be delineated elsewhere.

References

[1] K. Balakrishna Reddy and P. Subrahmanyam, *Extensions of Krasnoselskij's and Matkowski's fixed point theorems,* Funkc. Ekv., 24 (1981), 67-83.

[2] V. Berinde and M. Borcut, *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces,* Nonlinear Anal., 74 (2011), 4889-4897.

[3] T. G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications,* Nonlinear Anal., 65 (2006), 1379-1393.

[4] P. J. Bushell, *Hilbert's metric and positive contraction mappings in a Banach space,* Arch. Rational Mech. Anal., 52 (1973), 330-338.

[5] S. Czerwik, *A fixed point theorem for a system of multivalued transformations,* Proc. Amer. Math. Soc., 55 (1976), 136-139.

[6] J. Matkowski, *Some inequalities and a generalization of Banach's principle,* Bull. Acad. Pol. Sci. (Ser. Sci. Math. Astronom. Phys.), 21 (1973), 323-324.

[7] J.J. Nieto and R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations,* Order, 22 (2005), 223-239.

[8] D. O'Regan and A. Petruşel, *Fixed point theorems for generalized contractions in ordered metric spaces,* J. Math. Anal. Appl., 341 (2008), 1241-1252.

[9] A. I. Perov, *On the Cauchy problem for systems of ordinary differential equations,* (Russian), in "Approximate Methods for solving Differential Equations", pp. 115-134, Naukova Dumka. Kiev, 1964.

[10] T. Precupanu, *Linear Topological Spaces and Fundamentals of Convex Analysis,* (Romanian), Editura Academiei Române, Bucureşti, 1992.

[11] A. C. M. Ran and M. C. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations,* Proc. Amer. Math. Soc., 132 (2004), 1435-1443.

[12] I. A. Rus, *Generalized Contractions and Applications* Cluj University Press, Cluj-Napoca, 2001.

[13] M. Turinici, *Abstract comparison principles and multivariable Gronwall-Bellman inequalities,* J. Math. Anal. Appl., 117 (1986), 100-127.

"A. Myller" Mathematical Seminar; "A. I. Cuza" University; 700506 Iaşi, Romania
E-mail address: mturi@uaic.ro